Physics 200A — Fall [redacted] 2015

I.) Basic Lagrangian Mechanics (Read L&L: Chapt 1, 2)

→ Principle of <u>Least Action</u> / Hamilton's Principle — {Physics from Variational Principle}

If system of <u>point</u> <u>particles</u> (i.e. no internal degrees of freedom), such that

- <u>parametrized</u> <u>by</u> <u>generalized</u> <u>coordinates</u> $q_1, q_2, \dots q_s$, and <u>generalized</u> <u>velocities</u> $\dot{q}_1, \dot{q}_2, \dots \dot{q}_s$ where $q_i = q_i(t)$

  (n.b. time is <u>parameter</u>).

N.B. Generalized coordinates need <u>not</u> correspond to usual/familiar coordinate system. → useful for problem solving

- let system be described by a function $L = L(\underline{q}_i, \dot{\underline{q}}_i, t)$.

  explicit time dependence possible

  L is Lagrangian.

Then:

Trajectory $\underline{q}_1(t_1) \to \underline{q}_2(t_2)$ is one which minimizes action S

$$S = \int_{t_1}^{t_2} dt \; L(q_i, \dot{q}_i, t)$$

Action - functional of trajectory $q_i(t)$

- enables g.c.

i.e. trajectory selected by Principle of Least Action.

Observations:

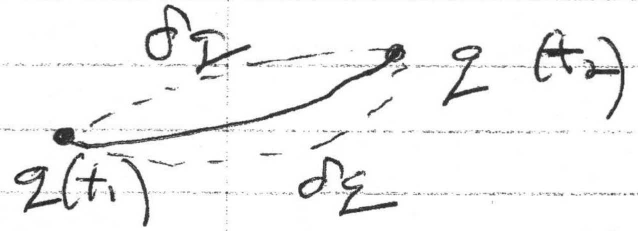
- * variational principle allows use of generalized coordinates

i.e. S minimized, parametrization independent. S not calculated.

- S it-self not determined. EOM is determined. Later, Hamilton-Jacobi determines S.
- PLA is energy method, as $L = T - U$ (+bd)

Now, consider variation of (necessary, not sufficient, for minimization) S:

$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial q} \delta q \right)$$



exchange deriv.

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial \dot{q}} \frac{d}{dt}(\delta q) + \frac{\partial L}{\partial q} \delta q \right)$$

$$= \frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\, \delta q(t) \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right)$$

but $\delta q(t_{1,2}) = 0$, i.e. on end points, so

$$\delta S = \int_{t_1}^{t_2} dt\, \delta q(t) \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right)$$

and $\delta S = 0$ for all δq, if:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$ — Lagrange's Equation

Observe:

- Lagrange's equations determine trajectory $q(t)$. I.C.'s will be needed for solution.

- L.E. are PDE

- Lagrangian invariant to df/dt addition (i.e. undetermined to $+df/dt$).

i.e. $S = \int dt\, L \rightarrow \int dt\,(L + df/dt)$

$$= S_0 + f(q(t_2)) - f(q(t_1))$$

$$\delta S = \delta S_0 + \delta(\underbrace{\Delta f}_{L_0})$$

$$\delta \Delta f = \Delta\, \partial f/\partial q\, \delta q \Big|_{t_2, t_1}$$

but $\delta q(t_{2,1}) = 0$

$\delta S = \delta S_0$ → no change in trajectory or physics.

Some obvious questions:

- is $q(t)$ obtained so a <u>minimum</u> of S?

⇒ Well, not really ... but is an extremum.

- what is L?

N.B. We know $L = T - V$

T → {kinetic energy}

V → {potential energy}

but what if we didnt?

⇒ Structure of L:

- generally, by <u>symmetry</u>

i.e. consider <u>free</u> particle:

- here <u>non-relativistic</u>

- space-time <u>homogeneity</u> ⇒ L cannot depend on $\underline{x}, t$; only on $\underline{v}$

- space-time <u>isotropy</u> ⇒ L depends on $v^2 = \underline{v} \cdot \underline{v}$, only (not $\underline{v}$ → has direction content

$\therefore \quad L = L(V^2)$

Aside: why $L = L(V^2)$ and
$\partial L/\partial V^2 = m/2$? why not $L(V^2) \sim \alpha (V^2)^2$?

$\Rightarrow$ Principle of (Galilean) Relativity:
$\Rightarrow$
For two frames of reference related by infinitesimal Galilean boost, trajectories must be same.

$\left\{ \begin{array}{l} \underline{r} = \underline{r}' + \underline{V}t \\ t = t', \quad \underline{v} = \underline{v}' + \underline{V} \\ \underline{V} = \text{const.} \end{array} \right.$

Approach: show $L((v+dv)^2)$ and $L(v^2)$ differ by df/dt
if [redacted] $\partial L/\partial V^2 = \text{const.}$

Check: $L[(v+dv)^2] - L(v^2)$

$$\cong L(v^2) + (2\underline{v}\cdot d\underline{v})\left(\frac{\partial L}{\partial v^2}\right) + \underset{\text{h.o.}}{d v^2 \left(\frac{\partial L}{\partial v^2}\right)} - L(v^2)$$

$$= (2\underline{v}\cdot d\underline{v})\frac{\partial L}{\partial v^2}$$

Now, if $\partial L / \partial v^2$ indep v^2 — i.e. constant:

$$L((v+\delta V)^2) - L(v^2) \cong m\underline{v} \cdot \delta \underline{V}$$

$\delta \underline{V}$ fixed parameter, so:

$$\cong \frac{d}{dt}(m\,\underline{x} \cdot \delta \underline{V})$$

$$= df/dt \rightarrow \text{irrelevant!}$$

so $\dfrac{\partial L}{\partial v^2} = \text{const} = \dfrac{m}{2}$, by correspondence.
makes L Galilean inv.

Note: $m > 0$ for minimum in S.

Thus: for free particle.

$$L = \tfrac{1}{2} m v^2 ,$$ by symmetry and Galilean relativity

↳ kinetic energy.

⇒ Newton's 1st Law.

N.B. To show:

$$\Delta L = \frac{m}{2}(\underline{v} + \underline{V})^2 - \frac{mv^2}{2}$$

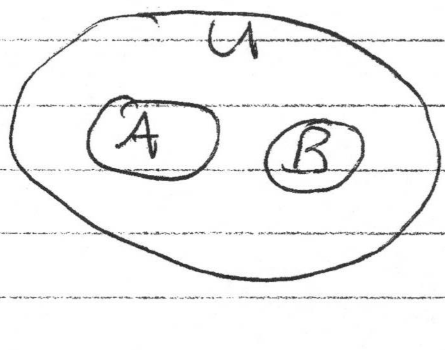
$$= \cancel{\frac{mv^2}{2}} + m\underline{v}\cdot\underline{V} + \frac{1}{2}m\underline{V}^2 - \cancel{\frac{mv^2}{2}}$$

$$= \frac{d}{dt}\underbrace{\left(m\underline{x}\cdot\underline{V} + \frac{1}{2}m\underline{V}^2 t\right)}_{f}$$

So : $L_{\text{free particle}} = mv^2/2$ → see 8a

– For interacting particles, i.e. not free:

→ useful to introduce concept of open, closed system.



$U \equiv$ universe
A, B → systems

systems:

closed → non-interacting
open → interacting

if U formed by two, closed subsystems A, B (i.e. 2 free particles)

K.E. for standard coords:

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \quad \text{Cartesian}$$

$$T = 1/2\, m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2), \quad \text{Cylindrical}$$

$$T = 1/2\, m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin\theta^2 \dot{\phi}^2), \quad \text{spherical}$$

n.b.:

$$d\ell^2 = dx^2 + dy^2 + dz^2$$

$$d\ell^2 = dr^2 + r^2 d\theta^2 + dz^2$$

$$d\ell^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

$$L_U = L_A + L_B \Rightarrow$$

Lagrangians for closed sub-systems additive $\Rightarrow$ i.e. L for 2 systems together must asymptote to that for sum of L's for individual systems, at large separation.

[$\rightarrow$ Consider 2 particles $\Rightarrow$ must go to 2 indiv. free particles.]

Now, in non-relativistic limit:

For system of interacting particles which is closed, Lagrangian can be written as:

$$L = \sum_i \frac{m_i v_i^2}{2} + Q(\underline{r}_1, \underline{r}_2 \ldots)$$

interaction potential
$\Leftrightarrow$ function of coordinates only

n.b.: $- \; v \ll c \quad (c \rightarrow \infty)$

$\infty \Rightarrow$ particle "feels" effect of neighbor at retarded time

$- \; r\left(t - \frac{|\Delta r|}{c}\right) \rightarrow r(t) \Rightarrow$ instantaneous $c \rightarrow \infty$

$$\frac{d}{dt} p_1 - \frac{\partial Q}{\partial r_1}(r_1, r_2, t) = 0$$

$$r_2 = r_2\left(t - \frac{|r_1 - r_2|}{c}\right) \xrightarrow[c \to \infty]{} r_2(t)$$

→ Now, in event that $q_i = x_i$ (generalized coordinates are Cartesian coordinates),

know Lagrange's equations must reduce to Newton's Laws.

(Correspondence argument for Q, L).

L. E.:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underline{v}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

⇒

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underline{v}_i}\right) - \frac{\partial Q}{\partial \underline{x}_i} = 0$$

$\overset{so}{=}$

$$\frac{d}{dt}\left(\underline{p}_i\right) - \frac{\partial Q}{\partial \underline{x}_i} = 0$$

$\overset{so}{=}$ $Q = -V$

$$\boxed{L = T - U}$$

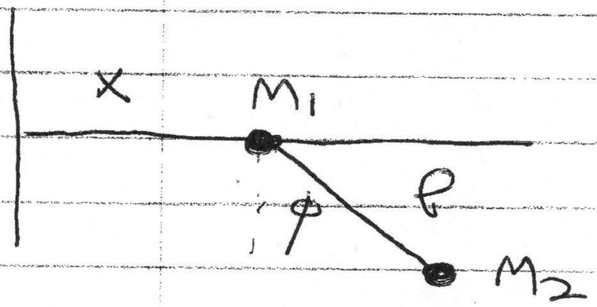
also natural to coin terminology:

$q_i \equiv$ generalized coordinate

$p_i = \frac{\partial L}{\partial \dot{q}_i} \equiv$ generalized momentum

Examples:

i) (Trivial)



Pendulum attached to freely sliding M_1

computing energies:

$m_1 \Rightarrow \quad T_1 = \frac{1}{2} m_1 \dot{x}^2$ $\qquad$ GC: x, ϕ

$U = 0$

$m_2 \Rightarrow \quad T_2 = \frac{1}{2} m_2 \left[\dot{x}^2 + \dot{y}^2 \right]$

$= \frac{1}{2} m_2 \left[\left(\dot{\overline{x + \ell \sin\phi}}\right)^2 + \left(\dot{\overline{\ell \cos\phi}}\right)^2 \right]$

$U = mg\ell(1 - \cos\phi)$

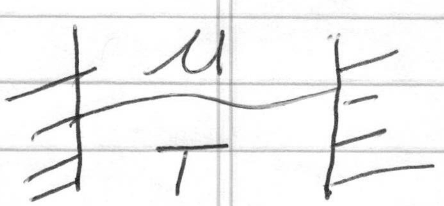
$$L = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \Big[\dot{x}^2 + 2 x \ell \dot{\phi} \cos\phi + \ell^2 \cos^2\phi \, \dot{\phi}^2 + \ell^2 \sin^2\phi \, \dot{\phi}^2 \Big] - mg\ell(1-\cos\phi)$$

$$L = \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + m_2 \Big[\frac{\ell^2}{2}\dot{\phi}^2 + \underbrace{x \ell \dot{\phi} \cos\phi}_{\text{coupling}} \Big] + mg\ell\cos\phi \cancel{- mg\ell}$$

and EOM for x, ϕ follow.

ii.) Non-trivial

Derive NL string equation with tension T, mass-per-length μ (1D):



- end point fixed
  $\delta y(0) = \delta y(L) = 0$
  all t
- $\delta y(x, t_2, t_1) = 0$
  (known initial, final conformation)

$$\sum_i = \int dx$$ → x, like t, is a parameter

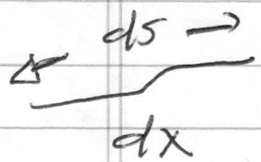
so

$$S = \int_{t_1}^{t_2} dt \int_0^L dx\, \mathcal{L}$$

$\Downarrow$
Lagrangian density

G.C.: $y(x,t)$

$$U = \int_0^L ds\, T - \int_0^L dx\, T$$



net extension or 'plucking' of string

$$U = \int_0^L dx\,\left(ds/dx - 1\right) T$$

$$ds^2 = dx^2 + dy^2$$
$$= dx^2\left(1 + (dy/dx)^2\right)$$

$$\mathcal{L} = \tfrac{1}{2}\mu\left(\partial_t y(x,t)\right)^2 - \left[T\left(1 + (dy/dx)^2\right)^{1/2} - T\right]$$

const

So

$$\delta S = \int dt \int_0^L \delta\left\{ \frac{\mu}{2}(\partial_t y)^2 - T\left(1 + (\partial_x y)^2\right)^{1/2} \right\}$$

$$= \int_{t_1}^{t_2} dt \int_0^L \left\{ \frac{\partial \mathcal{L}}{\partial y_t}\,\delta y_t + \frac{\partial \mathcal{L}}{\partial y_x}\,\delta y_x \right\}$$

i.c. fixed

$$= \int_0^L dx\, \frac{\partial \mathcal{L}}{\partial y_t}\,\delta y \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \int_0^L dx\, \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right)$$

$$+ \int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial y_x}\,\delta y \Big|_0^L dt - \int_{t_1}^{t_2}\int_0^L \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right)$$

e.p. fixed

so $\delta S = 0$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) = 0}$$

Lagrange equation

Now, $$\frac{\partial \mathcal{L}}{\partial y_t} = \mu y_t$$

$$\frac{\partial \mathcal{L}}{\partial y_x} = \frac{-T\,\partial y/\partial x}{\left[1 + (\partial y/\partial x)^2\right]^{1/2}}$$

so

$$\frac{d}{dt}\left(\mu y_t\right) - \frac{d}{dx}\left(\frac{T\,\partial y/\partial x}{\left[1 + (\partial y/\partial x)^2\right]^{1/2}}\right) = 0$$

- nonlinear equation for waves on clamped string.
- x, t param $\rightarrow$ ignore d vs ∂ distinction.

- so, linearizing : T, μ const

$|\partial y/\partial x| \ll 1$ → weak slope approximation

⇒

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$